Bessel Functions

1 A Special Class of Equations

Consider

$$(1 + R_M x^M)\frac{d^2y}{dx^2} + \frac{1}{x}(P_0 + P_M x^M)\frac{dy}{dx} + \frac{1}{x^2}(Q_0 + Q_M x^M)y = 0 \tag{1}$$

where, M is a positive integer. This equation is a special case of

$$\mathcal{L}y \equiv R(x)\frac{d^2y}{dx^2} + \frac{1}{x}P(x)\frac{dy}{dx} + \frac{1}{x^2}Q(x)y = 0 \tag{2}$$

since $R(x) = 1 + R_M x^M$, $P(x) = P_0 + P_M x^M$ and $Q(x) = Q_0 + Q_M x^M$. Let

$$y(x) = \sum_{k=0}^{\infty} a_k x^{k+s} \tag{3}$$

The Indicial Equation should be the same as before, i.e.,

$$f(s) = s^2 + (P_0 - 1)s + Q_0 = 0 \tag{4}$$

For $k = 1, 2, 3, \cdots$,

$$f(s+k)a_k + \sum_{i=1}^{k} g_i(s+k)a_{k-i} = 0 \tag{5}$$

But

$$g_i(s) = R_i(s-i)(s-i-1) + P_i(s-i) + Q_i \tag{6}$$

for $i = 1, 2, 3, \cdots$. Thus,

$$g_i(s) = 0 \tag{7}$$

for $i = 1, 2, 3, \cdots, M-1, M+1, \cdots$, and

$$g_M(s) \neq 0 \tag{8}$$

When $k = 1$,

$$f(s+1)a_1 + g_1(s+1)a_0 = 0 \qquad \Rightarrow a_1 = 0 \tag{9}$$

When $k = 2$,

$$f(s+2)a_2 + \sum_{i=1}^{2} g_i(s+2)a_{2-i} = 0 \qquad \Rightarrow a_2 = 0 \tag{10}$$

Thus,

$$a_1 = a_2 = \cdots = a_{M-1} = 0 \tag{11}$$

However, when $k = M$

$$f(s+M)a_M + \sum_{i=1}^{M} g_i(s+M)a_{k-i} = f(s+M)a_M + g_M(s+M)a_0 = 0$$

$$a_M = -\frac{g_M(s+M)}{f(s+M)}a_0 \tag{12}$$

When $k = M+1$,

$$f(s+M+1)a_{M+1} + g_M(s+M+1)a_1 = f(s+M+1)a_{M+1} = 0$$

$$a_{M+1} = 0 \tag{13}$$

Similarly,

$$a_{M+1} = a_{M+2} = \cdots = a_{2M-1} = 0 \tag{14}$$

When $k = 2M$,

$$f(s+2M)a_{2M} + g_M(s+2M)a_M = 0$$

$$a_{2M} = -\frac{g_M(s+2M)}{f(s+2M)}a_M = \frac{g_M(s+2M)}{f(s+2M)}\frac{g_M(s+M)}{f(s+M)}a_0 \tag{15}$$

Thus,

$$a_{nM+1} = a_{nM+2} = \cdots = a_{(n+1)M-1} = 0 \tag{16}$$

and

$$a_{(n+1)M} = -\frac{g_M(s+nM+M)}{f(s+nM+M)}a_{nM} \tag{17}$$

where $n = 0, 1, 2, \cdots$.

Therefore,

$$y(x) = \sum_{n=0}^{\infty} a_{nM}x^{nM+s} = \sum_{n=0}^{\infty} B_n x^{nM+s} \tag{18}$$

The exceptional cases will be

- $s_1 = s_2$, or
- $s_1 - s_2 = nM$, where n is a positive integer.

2 A Summary of Bessel Functions

The Bessel function satisfy

$$x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}+(x^2-p^2)y=0 \tag{19}$$

where $p \geq 0$.

- First Solution

$$J_p(x)=\sum_{k=0}^{\infty}\frac{(-1)^k\,(x/2)^{2k+p}}{\Gamma(k+p+1)k!} \tag{20}$$

- Second Solution

  1. $p \neq 0$ and p is not an integer

  $$J_{-p}(x)=\sum_{k=0}^{\infty}\frac{(-1)^k\,(x/2)^{2k-p}}{\Gamma(k-p+1)k!} \tag{21}$$

  2. $p = 0$

  $$Y_0(x)=\frac{2}{\pi}\left[\left(\ln\frac{x}{2}+\gamma\right)J_0(x)-\sum_{k=1}^{\infty}(-1)^k\varphi(k)\frac{(x/2)^{2k}}{(k!)^2}\right] \tag{22}$$

  where

  $$\varphi(k)=\sum_{m=1}^{k}\frac{1}{m} \tag{23}$$

  $$\gamma(\text{ Euler constant })=\lim_{k\to\infty}\left[\varphi(k)-\ln k\right]=0.57721566\cdots \tag{24}$$

  3. $p = \ell =$ a positive integer

  $$Y_\ell(x)=\frac{2}{\pi}\left\{\left(\ln\frac{x}{2}+\gamma\right)J_\ell(x)-\frac{1}{2}\sum_{k=0}^{\ell-1}\frac{(\ell-k-1)!(x/2)^{2k-\ell}}{k!}\right.$$
  $$\left.-\frac{1}{2}\sum_{k=0}^{\infty}(-1)^k\left[\varphi(k)+\varphi(k+\ell)\right]\frac{(x/2)^{2k+\ell}}{k!(\ell+k)!}\right\} \tag{25}$$

3 Origins of Bessel Functions

Solutions of the differential equation

$$x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2 - p^2)y = 0 \tag{26}$$

or

$$x\frac{d}{dx}\left(x\frac{dy}{dx}\right) + (x^2 - p^2)y = 0 \tag{27}$$

are known as *Bessel functions of order p*, where p is real and non-negative.

Since $M = 2$, the solution is of the following form:

$$y(x) = \sum_{k=0}^{\infty} B_k x^{2k+s} \tag{28}$$

After substituting this into equation (26), the resulting indicial equation yields:

$$s_1 = p \qquad s_2 = -p \tag{29}$$

Thus, the exceptional cases occur only when

- $s_1 = s_2 = 0$, i.e., $p = 0$.
- $s_1 - s_2 = 2p = 2\ell$, i.e., $p = \ell =$ a positive integer.

We can nonetheless always obtain the first solution for $s_1 = p$:

$$y_1(x) = B_0 \left[x^p + \sum_{k=1}^{\infty} \frac{(-1)^k x^{2k+p}}{2^{2k} k!(1+p)(2+p)\cdots(k+p)} \right] \tag{30}$$

According to the definition of gamma function, this solution can be written as

$$\begin{aligned} y_1(x) &= B_0 \Gamma(1+p) \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+p}}{2^{2k}\Gamma(k+p+1)k!} \\ &= 2^p \Gamma(1+p) B_0 \sum_{k=0}^{\infty} \frac{(-1)^k (x/2)^{2k+p}}{\Gamma(k+p+1)k!} \end{aligned} \tag{31}$$

Here, let us define

$$J_p(x) = \sum_{k=0}^{\infty} \frac{(-1)^k (x/2)^{2k+p}}{\Gamma(k+p+1)k!} \tag{32}$$

and $J_p(x)$ is known as the *Bessel function of the first kind, of order p.*

The second solution is presented in the sequel:

1. If $p \neq 0$ and is not an integer, a second solution is obtained by replacing p by $-p$ in the first solution, i.e.,

$$J_{-p}(x) = \sum_{k=0}^{\infty} \frac{(-1)^k (x/2)^{2k-p}}{\Gamma(k-p+1)k!} \tag{33}$$

Thus the complete solution of equation (26) is

$$y(x) = c_1 J_p(x) + c_2 J_{-p}(x) \tag{34}$$

However, if p is <u>*an integer*</u>, it can be shown that

$$J_{-p}(x) = (-1)^p J_p(x)$$

Thus, $J_p(x)$ and $J_{-p}(x)$ are *linearly independent* in this situation.

2. If $p = 0$, i.e., $s_1 = s_2 = 0$, then

$$y_2(x) = \left[\frac{\partial y(x,s)}{\partial s} \right]_{s=0} \tag{35}$$

The result of this derivation is

$$y_2(x) = B_0 \left[J_0(x) \ln x - \sum_{k=1}^{\infty} (-1)^k \varphi(k) \frac{(x/2)^{2k}}{(k!)^2} \right] \tag{36}$$

where

$$\varphi(k) = \sum_{m=1}^{k} \frac{1}{m}$$

Define

$$Y^{(0)}(x) = J_0(x) \ln x - \sum_{k=1}^{\infty} (-1)^k \varphi(k) \frac{(x/2)^{2k}}{(k!)^2} \tag{37}$$

Since equation (26) is linear, a linear combination of the above two solutions, i.e., $J_0(x)$ and $Y^{(0)}(x)$, is still a solution of the original differential equation. An alternative form of the second solution is thus more often used:

$$Y_0(x) = \frac{2}{\pi} \left[Y^{(0)} + (\gamma - \ln 2) J_0(x) \right]$$

$$= \frac{2}{\pi}\left[\left(\ln\frac{x}{2}+\gamma\right)J_0(x)+\sum_{k=1}^{\infty}(-1)^{k+1}\varphi(k)\frac{(x/2)^{2k}}{(k!)^2}\right] \tag{38}$$

where

$$\gamma = \lim_{k\to\infty}\left[\varphi(k)-\ln k\right] = 0.57721566490\cdots$$

This standard particular solution is called *Bessel function of the second kind of order zero* or *Neumann function of order zero.* This definition of the Bessel function of the 2nd kind is more convenient to use and thus is usually preferred, because of the fact that the behavior of the function $Y_0(x)$, for large value of x, is more comparable with the behavior of $J_0(x)$. Thus, the complete solution in this case is

$$y(x) = c_1 J_0(x) + c_2 Y_0(x) \tag{39}$$

3. $p = \ell =$ a positive integer.

$$y_2(x) = \left\{\frac{\partial}{\partial s}\left[(s+\ell)y(x,s)\right]\right\}_{s=-\ell} \tag{40}$$

The result

$$Y_\ell(x) = \frac{2}{\pi}\left\{\left(\ln\frac{x}{2}+\gamma\right)J_\ell(x)-\frac{1}{2}\sum_{k=0}^{\ell-1}\frac{(\ell-k-1)!(x/2)^{2k-\ell}}{k!}\right.$$
$$\left.-\frac{1}{2}\sum_{k=0}^{\infty}(-1)^k\left[\varphi(k)+\varphi(k+\ell)\right]\frac{(x/2)^{2k+\ell}}{k!(\ell+k)!}\right\} \tag{41}$$

Notice that the second solution is defined differently, depending on whether the order p is an integer or not. To provide uniformity of formalism and numerical tabulation. it is desirable to adopt a form of the second solution that is *valid for all values of the order.* The standard second solution $Y_p(x)$ defined for all p is

$$Y_p(x) = \frac{1}{\sin p\pi}\left[J_p(x)\cos p\pi - J_{-p}(x)\right]$$
$$Y_\ell(x) = \lim_{p\to\ell} Y_p(x)$$

Asymptotic Behaviors of Bessel Functions

$$J_0(0) = 1 \tag{42}$$

$$J_1(0) = J_2(0) = \cdots = 0 \tag{43}$$

$$J_0(\infty) = J_1(\infty) = \cdots = 0 \tag{44}$$

$$Y_0(0) = Y_1(0) = Y_2(0) = \cdots = -\infty \tag{45}$$

$$Y_0(\infty) = Y_1(\infty) = \cdots = 0 \tag{46}$$

Next, let $y = u/\sqrt{x}$, then substitute it into equation (26) to obtain

$$\frac{d^2u}{dx^2} + \left(1 - \frac{p^2 - 1/4}{x^2}\right) u = 0 \tag{47}$$

As $x \to \infty$,

$$1 \gg \frac{p^2 - 1/4}{x^2}$$

Thus,

$$\frac{d^2u}{dx^2} + u \simeq 0 \tag{48}$$

$$y \simeq \frac{1}{\sqrt{x}}(A\cos x + B\sin x) \tag{49}$$

It can be shown that, as $x \to \infty$,

$$J_p \sim \sqrt{\frac{2}{\pi x}} \cos\left(x - \alpha_p\right) \tag{50}$$

$$Y_p \sim \sqrt{\frac{2}{\pi x}} \sin\left(x - \alpha_p\right) \tag{51}$$

where,

$$\alpha_p = (2p + 1)\frac{\pi}{4}$$

On the other hand, if $p = 1/2$, then equation (47) can be reduced to

$$\frac{d^2u}{dx^2} + u = 0 \tag{52}$$

Thus, the complete solution is of the form

$$y = \frac{1}{\sqrt{x}}(A\cos x + B\sin x) \tag{53}$$

The standard solutions are

$$J_{1/2}(x) = \sqrt{\frac{2}{\pi x}} \sin x \tag{54}$$

$$J_{-1/2}(x) = \sqrt{\frac{2}{\pi x}} \cos x \tag{55}$$

4 Modified Bessel Functions

A slight variation of the standard Bessel equation (of order p) is the Bessel equation of order p with parameter α:

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + (\alpha^2x^2 - p^2)y = 0$$

This equation can be transformed into the standard form by substituting

$$t = \alpha x$$

Thus,

$$t^2\frac{d^2y}{dt^2} + t\frac{dy}{dt} + (t^2 - p^2)y = 0$$

The solution of this equation is

$$y = c_1J_p(t) + c_2J_{-p}(t) = c_1J_p(\alpha x) + c_2J_{-p}(\alpha x)$$

or

$$y = c_1J_p(t) + c_2Y_p(t) = c_1J_p(\alpha x) + c_2Y_p(\alpha x)$$

Next, let us consider

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} - (x^2 + p^2)y = 0 \tag{56}$$

which is also an alternative form of the Bessel's equation. If we substitute $t = ix$ into this equation, the resulting equation is

$$t^2\frac{d^2y}{dt^2} + t\frac{dy}{dt} + (t^2 - p^2)y = 0 \tag{57}$$

which is the same as the standard Bessel's equation.

- If p is not zero or a positive integer, the general solution is

$$y(x) = c_1J_p(ix) + c_2J_{-p}(ix) \tag{58}$$

- Otherwise,

$$y(x) = c_1J_\ell(ix) + c_2Y_\ell(ix) \tag{59}$$

In the above equations, it is necessary to use

$$J_p(ix) = \sum_{k=0}^{\infty} \frac{(-1)^k}{\Gamma(k+p+1)k!}(\frac{ix}{2})^{2k+p} = i^p \sum_{k=0}^{\infty} \frac{1}{\Gamma(k+p+1)k!}(\frac{x}{2})^{2k+p} \tag{60}$$

Define

$$I_p(x) = \sum_{k=0}^{\infty} \frac{1}{\Gamma(k+p+1)k!}(\frac{x}{2})^{2k+p} = i^{-p} J_p(ix) \tag{61}$$

where $I_p(x)$ is referred to as the *modified Bessel function of the first kind of order p*. Thus, the complete solution for non-zero and non-integer p is

$$y(x) = c_1 I_p(x) + c_2 I_{-p}(x) \tag{62}$$

If $p = \ell =$ a non-negative integer, the second solution can be redefined as

$$K_\ell(x) = \frac{\pi}{2} i^{\ell+1} \left[J_\ell(ix) + iY_\ell(ix) \right] \tag{63}$$

where $K_\ell(x)$ is referred to as the *modified Bessel function of the second kind of order ℓ*.

Asymptotic Behavior of $I_p(x)$ and $K_p(x)$ as $x \to \infty$

$$I_p(x) \sim \frac{e^x}{\sqrt{2\pi x}} \tag{64}$$

$$K_p(x) \sim \frac{e^{-x}}{\sqrt{2x/\pi}} \tag{65}$$

5 Properties of Bessel Functions

- For small values of x, i.e., as $x \to 0$,

$$J_p(x) \sim \frac{1}{2^p \Gamma(p+1)} x^p \tag{66}$$

$$J_{-p}(x) \sim \frac{2^p}{\Gamma(-p+1)} x^{-p} \qquad (p \neq \ell) \tag{67}$$

$$Y_p(x) \sim -\frac{2^p (p-1)!}{\pi} x^{-p} \qquad (p \neq 0) \tag{68}$$

$$Y_0(x) \sim \frac{2}{\pi} \ln x \tag{69}$$

$$I_p(x) \sim \frac{1}{2^p \Gamma(p+1)} x^p \tag{70}$$

$$I_{-p}(x) \sim \frac{2^p}{\Gamma(-p+1)} x^{-p} \qquad (p \neq \ell) \tag{71}$$

$$K_p(x) \sim 2^{p-1}(p-1)!x^{-p} \qquad (p \neq 0) \tag{72}$$

$$K_0(x) \sim -\ln x \tag{73}$$

Notice that only $J_p(x)$ and $I_p(x)$ are finite at $x = 0$ for $p \geq 0$.

- Differential properties:

$$\frac{d}{dx}\left[x^p y_p(\alpha x)\right] = \begin{cases} \alpha x^p y_{p-1}(\alpha x) & \text{for } y \equiv J, Y, I \\ -\alpha x^p y_{p-1}(\alpha x) & \text{for } y \equiv K \end{cases} \tag{74}$$

$$\frac{d}{dx}\left[x^{-p} y_p(\alpha x)\right] = \begin{cases} -\alpha x^{-p} y_{p+1}(\alpha x) & \text{for } y \equiv J, Y, K \\ \alpha x^{-p} y_{p+1}(\alpha x) & \text{for } y \equiv I \end{cases} \tag{75}$$

These formulas are established for J_p and Y_p by considering their series definitions, and for the remaining functions by considering their definitions in terms of J_p and Y_p.

From equation (74),

$$\frac{dy_p(\alpha x)}{dx} = \begin{cases} \alpha y_{p-1}(\alpha x) - (p/x) y_p(\alpha x) & \text{for } y \equiv J, Y, I \\ -\alpha y_{p-1}(\alpha x) - (p/x) y_p(\alpha x) & \text{for } y \equiv K \end{cases} \tag{76}$$

From equation (75),

$$\frac{dy_p(\alpha x)}{dx} = \begin{cases} -\alpha y_{p+1}(\alpha x) + (p/x) y_p(\alpha x) & \text{for } y \equiv J, Y, K \\ \alpha y_{p+1}(\alpha x) + (p/x) y_p(\alpha x) & \text{for } y \equiv I \end{cases} \tag{77}$$

– For J and Y,

  By adding equations (76) and (77) and then dividing the result by 2, one can obtain

$$\frac{dy_p(\alpha x)}{dx} = \frac{\alpha}{2}\left[y_{p-1}(\alpha x) - y_{p+1}(\alpha x)\right] \tag{78}$$

  By subtracting equation (77) from equation (76) and then dividing the result by 2, one can obtain

$$y_{p-1}(\alpha x) + y_{p+1}(\alpha x) = \frac{2p}{\alpha x} y_p(\alpha x) \tag{79}$$

– For I,

$$\frac{dI_p(\alpha x)}{dx} = \frac{\alpha}{2}\left[I_{p-1}(\alpha x) + I_{p+1}(\alpha x)\right] \tag{80}$$

$$I_{p-1}(\alpha x) - I_{p+1}(\alpha x) = \frac{2p}{\alpha x} I_p(\alpha x) \tag{81}$$

– For K,

$$\frac{dK_p(\alpha x)}{dx} = -\frac{\alpha}{2}\left[K_{p-1}(\alpha x) + K_{p+1}(\alpha x)\right] \tag{82}$$

$$K_{p-1}(\alpha x) - K_{p+1}(\alpha x) = -\frac{2p}{\alpha x} K_p(\alpha x) \tag{83}$$

[**Example**]

Using the table of J_0 and J_1 to integrate

$$I = \int_1^2 x^{-3} J_4(x) dx$$

[**Solution**]

From (75) with $p = 3$ and $\alpha = 1$ we obtain

$$\int_1^2 x^{-3} J_4(x) dx = -x^{-3} J_3(x)\Big|_{x=1}^{x=2}$$

By (79) with $p = 2$ and $\alpha = 1$ we have

$$J_1(x) + J_3(x) = \frac{4}{x} J_2(x) \qquad \Rightarrow \qquad J_3(x) = \frac{4}{x} J_2(x) - J_1(x)$$

Again by (79) with $p = 1$ and $\alpha = 1$,

$$J_0(x) + J_2(x) = \frac{2}{x} J_1(x) \qquad \Rightarrow \qquad J_2(x) = \frac{2}{x} J_1(x) - J_0(x)$$

Thus,

$$J_3(x) = \frac{4}{x}\left[\frac{2}{x} J_1(x) - J_0(x)\right] - J_1(x) = \left(\frac{8}{x^2} - 1\right) J_1(x) - \frac{4}{x} J_0(x)$$

From Table A1,

$$J_1(2) = 0.5767; \qquad J_0(2) = 0.2239 \qquad \Rightarrow \qquad J_3(2) = J_1(2) - 2J_0(2) = 0.1289$$

$$J_1(1) = 0.4401; \qquad J_0(1) = 0.7652 \qquad \Rightarrow \qquad J_3(1) = 7J_1(1) - 4J_0(1) = 0.0199$$

$$I = -\left[\frac{1}{8} J_3(2) - \frac{1}{1} J_3(1)\right] = 0.0038$$

[Example]

Determine $J_{\frac{3}{2}}(x)$ and $J_{-\frac{3}{2}}(x)$

[Solution]

From (79), (54) and (55) and let $p = 1/2$ and $\alpha = 1$, we get

$$J_{\frac{3}{2}}(x) = \frac{1}{x}J_{\frac{1}{2}} - J_{-\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}}\left(\frac{\sin x}{x} - \cos x\right)$$

From (79), (54) and (55), and let $p = -1/2$ and $\alpha = 1$, we can obtain

$$J_{-\frac{3}{2}}(x) = -\frac{1}{x}J_{-\frac{1}{2}} - J_{\frac{1}{2}}(x) = -\sqrt{\frac{2}{\pi x}}\left(\frac{\cos x}{x} + \sin x\right)$$

6 Differential Equations Satisfied by Bessel Functions

Given that the solution of

$$X^2\frac{d^2Y}{dX^2} + X\frac{dY}{dX} + (X^2 - p^2)Y = 0 \tag{84}$$

can be written in the form

$$Y = Z_p(X) = \begin{cases} c_1J_p(X) + c_2J_{-p}(X) & p \neq 0 \text{ and } p \neq \text{ integer} \\ c_1J_p(X) + c_2Y_p(X) & p = 0 \text{ or } p = \text{ integer} \end{cases} \tag{85}$$

Let

$$X = f(x) \qquad Y = \frac{y}{g(x)}$$

Note that

$$\frac{d}{dX} = \frac{d}{dx}\frac{dx}{dX} = \frac{1}{f'(x)}\frac{d}{dx} \tag{86}$$

Substitution into equation (84) yields

$$f\frac{d}{dx}\left[\frac{1}{f'}\frac{d}{dx}\left(\frac{y}{g}\right)\right] + \frac{d}{dx}\left(\frac{y}{g}\right) + \frac{f'}{f}(f^2 - p^2)\frac{y}{g} = 0 \tag{87}$$

The solution of this equation is

$$y = g(x)Z_p[f(x)] \tag{88}$$

In particular, if we select

$$f(x) = \frac{\sqrt{d}}{s} x^s \tag{89}$$

$$g(x) = x^{(1-a)/2} \exp\left[-\frac{b}{r} x^r\right] \tag{90}$$

$$p = \frac{1}{s} \sqrt{\left(\frac{1-a}{2}\right)^2 - c} \tag{91}$$

where a, b, c, d , r and s are constants. Then equation (87) can be transformed to

$$x^2 \frac{dy^2}{dx^2} + x(a + 2bx^r) \frac{dy}{dx} + \left[c + dx^{2s} - b(1 - a - r)x^r + b^2 x^{2r}\right] y = 0 \tag{92}$$

The solution of this equation is

$$y = g(x) Z_p[f(x)] = x^{\frac{1-a}{2}} \exp\left[-\frac{b}{r} x^r\right] Z_p \left[\frac{\sqrt{d}}{s} x^s\right] \tag{93}$$

where

$$p = \frac{1}{s} \sqrt{\left(\frac{1-a}{2}\right)^2 - c}$$

Thus, if it is possible to identify a particular second order differential equation with equation (92) by suitably choosing the constants a, b, c, d , r and s, the solution is immediately given in terms of Bessel function of order p.

[Example] $x^2 y'' + xy' + (\lambda^2 x^2 - p^2) y = 0 \qquad (\lambda \neq 0)$

Since $a = 1$, $b = 0$, $c = -p^2$, $d = \lambda^2$, $r = r$ and $s = 1$, then

$$p = \frac{1}{1} \left[\left(\frac{1-1}{2}\right)^2 - (-p^2)\right]^{0.5} = p$$

Thus,

$$y = Z_p(\lambda x)$$

[Example] $x^2 y'' + xy' - (\lambda^2 x^2 + p^2) y = 0 \qquad (\lambda \neq 0)$

Since $a = 1$, $b = 0$, $c = -p^2$, $d = -\lambda^2$, $r = r$ and $s = 1$, then

$$p = \frac{1}{1} \left[\left(\frac{1-1}{2}\right)^2 - (-p^2)\right]^{0.5} = p$$

Thus,

$$y = Z_p(i\lambda x)$$

[**Exercise**] $y'' + 3x^5 y = 0$

Ans:

$$y = \sqrt{x} Z_{\frac{1}{7}} \left(\frac{2\sqrt{3}}{7} x^{\frac{7}{2}} \right)$$

[**Exercise**] $y'' + 5x^4 y = 0$

Ans:

$$y = \sqrt{x} Z_{\frac{1}{6}} \left(\frac{\sqrt{5}}{3} x^3 \right)$$